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# Grinding with classification via geometric partition models

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## Abstract

A grinding process with classification is considered. An integral grinding equation connecting the final particle size distribution function to the particle size distribution function before the grinding process is studied. Geometric partition models are used to obtain the breakage function. The results are compared with experimental data.

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## 1. Introduction

Grinding systems used to produce powder coatings [1] usually contain grinders and classifiers [2]. The grinder is responsible for the particle size reduction and the classifier separates small particles and takes them out from the grinder.

In this paper, we derive an integral grinding equation connecting the grinding system ‘output’, the final particle size distribution, to the ‘input’, the particle size distribution before the grinding process. In the case of an ideal classifier, the integral grinding equation is a simple consequence of the well-known batch grinding equation [3]. If the classifier is not ideal, the integral grinding equation can be deduced directly from simple probabilistic considerations. The kernel of the integral grinding equation contains the so-called breakage function. We develop a geometric partition model approach to calculate this function. Moreover, the model involves particle shape, an important characteristic affecting many powder properties [4]. As a result, we get a mathematical model of a grinder with a classifier depending on three parameters. These parameters can be adjusted in order to model given a grinding system. We show this for a laboratory mill and for two different materials used in powder coating technology.

This mathematical model can be used to predict the results of grinding, to construct grinding systems with desired properties, and to improve the particle size measurement.

The paper is organized as follows. In section 2 the integral grinding equation is derived from the batch grinding equation in the case of an ideal classifier. In section 3 the same equation is derived using another approach suitable also for non-ideal classifiers. Section 4 is devoted to geometric partition models. In section 5, the partition operator is studied from the moment analysis point of view. Section 6 contains a model example. In section 7, the results of comparison with experimental data are presented. Finally, the last section contains concluding remarks.

## 2. Integral grinding equation

The fundamental equation of fragmentation, known as batch grinding equation, has the form

$$\frac{\partial f(V, t)}{\partial t} = -s(V)f(V, t) + \int_V^\infty b(V, W)s(W)f(W, t) dW, \quad (1)$$

where  $f(V, t)$  is the size density function at the moment  $t$ ,  $b(V, W)$  is the breakage function, giving the fraction of particles with volumes in the range  $[V, V + dV]$  obtained by breakage of a particle of volume  $W$ , and  $s(V)$  is the breakage rate of particles of volume  $V$  (see [3], for example). Equation (1) describes a grinding process without separation and is in a good agreement with the experimental data [5]. During this process the particles do not leave the grinder. Another type of grinding includes a separation process. A special device, known as a classifier, separates small particles with the volume  $V < \alpha$  and takes them out from the grinder. In other words  $s(V) = 0$ , whenever  $V < \alpha$ . We use (1) to deduce a new integral equation connecting the input and output particle size density functions.

Integrating equation (1) and setting

$$g(V) = s(V) \int_0^\infty f(V, t) dt,$$

we get

$$\lim_{t \rightarrow \infty} f(V, t) + g(V) = \int_V^\infty b(V, W)g(W) dW + f(V, 0).$$

All known breakage functions have the form

$$b(V, W) = \psi\left(\frac{V}{W}\right) \frac{V}{W^2}.$$

(This structure of  $b$  was confirmed by numerous experiments [5].) After the change of variables  $\eta = V/W$  in the integral, we obtain

$$\lim_{t \rightarrow \infty} f(V, t) + g(V) = \int_0^1 g\left(\frac{V}{\eta}\right) \psi(\eta) d\eta + f(V, 0). \quad (2)$$

Then we have

$$\lim_{t \rightarrow \infty} f(V, t) = \int_0^{V/\alpha} g\left(\frac{V}{\eta}\right) \psi(\eta) d\eta + f(V, 0), \quad V < \alpha. \quad (3)$$

Since  $\lim_{t \rightarrow \infty} f(V, t) = 0$  whenever  $V \geq \alpha$ , equation (2) implies

$$g(V) = \int_0^1 g\left(\frac{V}{\eta}\right) \psi(\eta) d\eta + f(V, 0), \quad V \geq \alpha. \quad (4)$$

This is, probably, the easiest way to derive the grinding equation, although the structure of partition operator must be postulated.

The batch grinding integro-differential equation describes distribution at any stage of grinding process and contains the breakage rate function  $s(V)$ . Theoretical or experimental determination of this function causes serious difficulties. Equations (3) and (4) do not contain  $s(V)$  and are more suitable to predict the final distribution, for any given feed distribution. Since system (3) and (4) is a consequence of (1), it does not contradict the experimental data.

### 3. Another way to deduce the integral grinding equation

Consider a set of particles. Let  $\nu(V) dV$  be the number of particles with the volumes in the interval  $[V, V + dV]$ . The density function  $f$  for the random value  $V$  (the particle volume) is given by

$$f(V) = \frac{\nu(V)}{\int_0^\infty \nu(V) dV}. \quad (5)$$

#### 3.1. Partition operator

Suppose that a particle with the volume  $V$  is divided into  $n$  parts with the volumes  $\xi_k V, k = \overline{1, n}$ , where

$$\bar{\xi} = (\xi_1, \dots, \xi_n) \in \Xi = \{(\xi_1, \dots, \xi_n) \mid \xi_k \geq 0, \xi_1 + \dots + \xi_n = 1\}.$$

Let the probability of getting particles with the volumes belonging to the intervals  $[\xi_k V, (\xi_k + d\xi_k)V]$  be  $\phi(\bar{\xi}) d\bar{\xi}$ , where  $\phi : \Xi \rightarrow R$  is a symmetric density function. Denote by  $\mathcal{P}(f)$  the density function for the particle volume after the partition. Obviously

$$\mathcal{P}(f)(V) dV = \frac{\mathcal{P}(\nu)(V) dV}{\int_0^\infty \mathcal{P}(\nu)(V) dV}, \quad (6)$$

where  $\mathcal{P}(\nu)(V) dV$  stands for the number of particles with the volumes in the interval  $[V, V + dV]$  after the partition. Observe that the number  $\mathcal{P}(\nu)(V) dV$  can be written as a superficial integral

$$\mathcal{P}(\nu)(V) dV = \int_{\Xi} \sum_{k=1}^n \nu\left(\frac{V}{\xi_k}\right) d\left(\frac{V}{\xi_k}\right) \phi(\bar{\xi}) dS_{\bar{\xi}}.$$

Dividing this equality by

$$\int_0^\infty \mathcal{P}(\nu)(V) dV = n \int_0^\infty \nu(V) dV$$

and invoking (5) and (6), we get

$$\mathcal{P}(f)(V) = \frac{1}{n} \int_{\Xi} \sum_{k=1}^n f\left(\frac{V}{\xi_k}\right) d\left(\frac{V}{\xi_k}\right) \phi(\bar{\xi}) dS_{\bar{\xi}}.$$

Since the function  $\phi$  is symmetric, we obtain

$$\mathcal{P}(f)(V) = \int_0^1 f\left(\frac{V}{\eta}\right) \psi(\eta) d\eta, \quad (7)$$

where

$$\psi(\eta) = \frac{1}{\eta} \int_{\Xi_\eta} \phi(\eta, \xi_2, \dots, \xi_n) dS_{\bar{\xi}},$$

and

$$\Xi_\eta = \{\bar{\xi} = (\xi_2, \dots, \xi_n) \mid \xi_k \geq 0, \eta + \xi_2 + \dots + \xi_n = 1\}.$$

Obviously

$$\int_0^1 \eta \psi(\eta) d\eta = \int_{\Xi} \phi(\bar{\xi}) dS_{\bar{\xi}} = 1. \quad (8)$$

Thus equation (7) gives a general form of the partition operator. The function  $\psi$  satisfying (8) can be found experimentally or derived theoretically from partition models.

### 3.2. Classifier

The grinding process can be modelled as a successive application of the partition operator given by (7):

$$f_{\text{out}} = \mathcal{P}^N(f_{\text{in}}),$$

where  $f_{\text{out}}$  and  $f_{\text{in}}$  stand for the final particle size density function and the initial particle size density function, respectively. During this process the particles do not leave the grinder.

To model the grinding process with separation, introduce a classifier operator  $\mathcal{C}_{\alpha,\sigma}$ ,  $\alpha$ ,  $\sigma > 0$ , defined by

$$\mathcal{C}_{\alpha,\sigma}(f)(V) = c_{\alpha,\sigma}(V)f(V),$$

where  $c_{\alpha,\sigma}$  is a distribution function describing the probability of the following event: a particle with the volume  $V$  leaves the grinder. This event depends on many small random variables. Therefore at first approximation it depends on the sum of these random variables and the corresponding probability distribution is the normal one:

$$c_{\alpha,\sigma}(V) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^V \exp\left(-\frac{(W-\alpha)^2}{2\sigma^2}\right) dW. \quad (9)$$

When  $\sigma$  goes to infinity, this distribution function tends to the distribution function corresponding to the ideal classifier considered in the previous section:

$$c_{\alpha}(V) = \begin{cases} 0, & V < \alpha, \\ 1, & V \geq \alpha. \end{cases}$$

The grinding process with classification can be represented in the following form

$$f_{\text{out}} = \sum_{k=0}^{\infty} (\mathcal{I} - \mathcal{C}_{\alpha,\sigma})(\mathcal{P} \circ \mathcal{C}_{\alpha,\sigma})^k(f_{\text{in}}),$$

where  $\mathcal{I}$  is the identity operator. If the separation takes place after more than one partition we have

$$f_{\text{out}} = \sum_{k=0}^{\infty} (\mathcal{I} - \mathcal{C}_{\alpha,\sigma})(\mathcal{P}^{\zeta} \circ \mathcal{C}_{\alpha,\sigma})^k(f_{\text{in}}), \quad (10)$$

here  $\zeta$  is the number of partitions before the separation. This formula can be used to model the grinding process if more than two particles result from the breakup of a particle.

### 3.3. Grinding equation

Equivalently the output can be written as

$$f_{\text{out}} = (\mathcal{I} - \mathcal{C}_{\alpha,\sigma})(g), \quad (11)$$

where

$$g = \sum_{k=0}^{\infty} (\mathcal{P} \circ \mathcal{C}_{\alpha,\sigma})^k(f_{\text{in}}). \quad (12)$$

Applying (formally) the operator  $(\mathcal{I} - \mathcal{P} \circ C_{\alpha,\sigma})$  to (12), we obtain the grinding equation

$$g = \mathcal{P} \circ C_{\alpha,\sigma}(g) + f_{\text{in}}. \quad (13)$$

Using (7) grinding equation (13) can be written in the integral form

$$g(V) = \int_0^1 c_{\alpha,\sigma} \left( \frac{V}{\eta} \right) g \left( \frac{V}{\eta} \right) \psi(\eta) d\eta + f_{\text{in}}(V).$$

In the ideal classifier case we get

$$g(V) = \int_0^{V/\alpha} g \left( \frac{V}{\eta} \right) \psi(\eta) d\eta + f_{\text{in}}(V). \quad (14)$$

From (11) we have  $f_{\text{out}}(V) = g(V)$ ,  $V < \alpha$ . Therefore (14) implies

$$f_{\text{out}}(V) = \int_0^{V/\alpha} g \left( \frac{V}{\eta} \right) \psi(\eta) d\eta + f_{\text{in}}(V), \quad V < \alpha. \quad (15)$$

It is easy to see that the integral in (15) depends only on the values  $g(V)$ , with  $V \in [\alpha, \infty[$ . Hence to find the output from (15) it suffices to solve integral equation (14) in the interval  $[\alpha, \infty[$ , that is, to solve the equation

$$g(V) = \int_0^1 g \left( \frac{V}{\eta} \right) \psi(\eta) d\eta + f_{\text{in}}(V), \quad V \geq \alpha. \quad (16)$$

Equalities (15) and (16) coincide with (3) and (4), respectively.

#### 4. Geometric partition models

The fragmentation of a particle in the grinder obviously depends on the particle shape. For the sake of simplicity assume that there exists only a finite number  $M$  of shapes and any particle of the shape  $m = \overline{1, M}$  is divided into at most two particles of shapes  $m' = \overline{1, M}$  and  $m'' = \overline{1, M}$ . Such a partition model can be obtained using some approximation rules. For example, if we have only spherical particles ( $M = 1$ ), then any particle is divided into two particles (obviously non-spherical) with volumes  $V_1$  and  $V_2$ . To form a one-shape partition model, we have to approximate the new particles by spheres with volumes  $V_1$  and  $V_2$ . The partition model can be completely artificial or based on a physical hypothesis. The shape set should be chosen rather simple, a finite number of polyhedrons or ellipsoids [6], for example. Consider one possible partition model. The powder is formed by ellipsoids of shapes  $m = \overline{1, M}$ . If the particle is sufficiently small, the geometry of the grinder is not important. Any fragmentation can be seen as a result of collision of a particle with an infinite rigid plane  $\Pi_0$ . An ellipsoid-shaped particle  $E$  is divided into two particles by a plane  $\Pi_0$  containing the normal vector to the plane  $\Pi_0$  at the point of collision  $\Pi_0 \cap E$ , and such that the area of the ellipse  $E \cap \Pi_0$  is minimal. This hypothesis is quite natural: in this case the energy needed to divide the particle is the minimal one. Each of the new particles is approximated by ellipsoids  $E'$  and  $E''$  of shapes  $m' = \overline{1, M}$  and  $m'' = \overline{1, M}$ , respectively. If  $V$  is the volume of the ellipsoid  $E$ , then the ellipsoids  $E'$  and  $E''$  have volumes  $V'$  and  $V''$  satisfying  $V = V' + V''$ . To model the grinding process, it suffices to consider only a finite number of possible orientations  $l = \overline{1, L}$ , of the particles with respect to the plane  $\Pi_0$ . For example, the normal vector to the plane  $\Pi_0$  at the point of collision  $\Pi_0 \cap E$  is parallel to one of the ellipsoid axes.

A geometric partition model can be described by a finite number of rules

$$\mathcal{R}(m, l) = (m'(m, l), m''(m, l), \gamma), \quad m = \overline{1, M}, l = \overline{1, L},$$

which establish a correspondence between a pair  $(m, l)$  (shape and orientation) and a pair of new shapes  $m'$  and  $m''$ , and the ratio of the volumes  $\gamma = V'/V''$ ,  $0 \leq V' \leq V''$ . Obviously

$$V' = \frac{\gamma}{1+\gamma} V \quad \text{and} \quad V'' = \frac{1}{1+\gamma} V. \quad (17)$$

If a particle of the shape  $m$  with the orientation  $l$  is not divided, we use the rule  $\mathcal{R}(m, l) = (0, m, 0)$ . A simple illustrative example of a geometric partition model is considered in section 6. If the number of shapes  $M$  and of orientations  $L$  are big enough, one can get a partition model close to reality.

Let  $v_m(V) dV$  be the number of particles of the shape  $m$  with the volumes in the interval  $[V, V + dV]$ . Consider the functions  $f_m$ ,  $m = \overline{1, M}$ , given by

$$f_m(V) = \frac{v_m(V)}{\sum_{n=1}^M \int_0^\infty v_n(V) dV}.$$

Put

$$\mathcal{P}(f_m)(V) = \frac{\mathcal{P}(v_m)(V)}{\sum_{n=1}^M \int_0^\infty \mathcal{P}(v_n)(V) dV},$$

where  $\mathcal{P}(v_m)(V) dV$  stands for the number of particles with the shape  $m$  and the volumes in the interval  $[V, V + dV]$ , after the partition. Assume that all orientations of the particles with respect to the plane  $\Pi_0$  are equally likely. This is a natural assumption in the case of isotropic materials. (If materials with a crystalline structure are considered, then it is necessary to introduce corresponding probabilities of the orientations.) Using (17) we have

$$\mathcal{P}(v_k)(V) dV = \frac{1}{L} \sum_{l=1}^L \sum_{(m, \theta) \in N(k, l)} v_m(\theta V) d(\theta V), \quad (18)$$

where

$$N(k, l) = \{(m, \theta) \mid \mathcal{R}(m, l) = (k, n, 1/(\theta - 1)) \text{ or } \mathcal{R}(m, l) = (n, k, \theta - 1)\}.$$

Set

$$\tau(m, l) = \begin{cases} 1, & m'(m, l) = 0, \\ 2, & m'(m, l) \neq 0. \end{cases}$$

Assume that there exists a number  $\tau$  satisfying

$$\tau = \frac{1}{L} \sum_{l=1}^L \tau(m, l), \quad m = \overline{1, M}. \quad (19)$$

For example, if all particles are divided into two parts, then this condition is satisfied and  $\tau = 2$ . Since

$$\sum_{n=1}^M \int_0^\infty \mathcal{P}(v_n)(V) dV = \tau \sum_{n=1}^M \int_0^\infty v_n(V) dV,$$

dividing (18) by

$$\sum_{n=1}^M \int_0^\infty \mathcal{P}(v_n)(W) dW dV,$$

we get

$$\mathcal{P}(f_k)(V) = \frac{1}{\tau L} \sum_{l=1}^L \sum_{(m, \theta) \in N(k, l)} \theta f_m(\theta V). \quad (20)$$

The density function  $f$  can be represented in the form

$$f(V) = \sum_{m=1}^M f_m(V).$$

The partition operator now takes the form

$$\mathcal{P}(f) = \sum_{m=1}^M \mathcal{P}(f_m), \quad (21)$$

where  $\mathcal{P}(f_m)$ ,  $m = \overline{1, M}$ , are given by (20).

## 5. Partition operator via geometric partition models

The partition operator given by (21) cannot be reduced to form (7). The study of moments helps to understand the relation between (21) and (7), to define fractional powers of the partition operator, and to clarify the mathematical mechanism of grinding process.

### 5.1. Moments

Set

$$\mu_s^{(m)} = \int_0^\infty V^s f_m(V) dV,$$

and

$$\mathcal{P}(\mu_s^{(m)}) = \int_0^\infty V^s \mathcal{P}(f_m(V)) dV, \quad m = \overline{1, M}, \quad s = 0, 1, \dots$$

From (20) we have

$$\mathcal{P}(\mu_s^{(k)}) = \frac{1}{\tau L} \sum_{l=1}^L \sum_{(m,\theta) \in N(k,l)} \theta^{-s} \mu_s^{(m)}, \quad k = \overline{1, M}, \quad s = 0, 1, \dots \quad (22)$$

Introducing vectors  $\bar{\mu}_s$  and  $\mathcal{P}(\bar{\mu}_s)$  with the components  $\mu_s^{(m)}$  and  $\mathcal{P}(\mu_s^{(m)})$ ,  $m = \overline{1, M}$ , respectively, equalities (22) can be written as

$$\mathcal{P}(\bar{\mu}_s) = P_s \bar{\mu}_s, \quad s = 0, 1, \dots, \quad (23)$$

where  $P_s$  is an  $M \times M$  matrix with the elements

$$(P_s)_{km} = \frac{1}{\tau L} \sum_{l=1}^L \sum_{\{\theta | (m,\theta) \in N(k,l)\}} \theta^{-s}.$$

Note that (19) implies

$$\sum_{k=1}^M (P_0)_{km} = \frac{1}{\tau L} \sum_{l=1}^L \tau(m, l) = 1, \quad m = \overline{1, M},$$

that is, the matrix  $P_0$  is stochastic.

Consider the moments of the density functions  $f$  and  $\mathcal{P}(f)$ :

$$\mu_s = \int_0^\infty V^s f(V) dV = \sum_{m=1}^M \mu_s^{(m)}, \quad s = 0, 1, \dots,$$



and

$$\mathcal{P}(\mu_s) = \int_0^\infty V^s \mathcal{P}(f)(V) dV, \quad s = 0, 1, \dots$$

If  $\mathcal{P}(f)$  is given by (7), then we have

$$\mathcal{P}(\mu_s) = \nu_{s+1} \mu_s, \quad s = 0, 1, \dots, \quad (24)$$

where

$$\nu_s = \int_0^\infty \eta^s \psi(\eta) d\eta, \quad s = 0, 1, \dots$$

This observation makes it possible to introduce fractional power  $\zeta$  of the partition operator as

$$\mathcal{P}^\zeta(\mu_s) = \nu_{s+1}^\zeta \mu_s, \quad s = 0, 1, \dots,$$

and to generalize (10) for all positive real  $\zeta$ . The corresponding function  $\psi$ , if it exists, is denoted by  $\psi_\zeta$ :

$$\mathcal{P}^\zeta(f) = \int_0^1 f\left(\frac{V}{\eta}\right) \psi_\zeta(\eta) d\eta. \quad (25)$$

On the other hand, if  $\mathcal{P}(f)$  is given by (21), then we obtain

$$\mathcal{P}(\mu_s) = \sum_{m=1}^M \mathcal{P}(\mu_s^{(m)}), \quad s = 0, 1, \dots, \quad (26)$$

where  $\mathcal{P}(\mu_s^{(m)})$ ,  $m = \overline{1, M}$ , are defined by (22) or, equivalently, by (23). Moment transformation (24) is a special case of (26). Indeed, if  $P_s \bar{\mu}_s = \nu_{s+1} \bar{\mu}_s$ ,  $s = 0, 1, \dots$ , then (24) and (26) coincide.

### 5.2. Partition operator in a multi-shape model

Assume that some power of matrix  $P_s$  is positive. Let  $\lambda_s$ ,  $s = 0, 1, \dots$ , be the dominant eigenvalues and  $\hat{\mu}_s$ ,  $s = 0, 1, \dots$ , be the corresponding positive eigenvectors of the matrices  $P_s$ ,  $s = 0, 1, \dots$ , [7]. Then for any positive vector  $\bar{\mu}$  the sequence  $\lambda_s^{-N} P_s^N \bar{\mu}$  tends to  $\hat{\mu}_s$ , as  $N$  goes to infinity [7]. This implies that

$$\sum_{m=1}^M \mathcal{P}^N(\mu_s^{(m)}) \approx \lambda_s^N \mu_s,$$

whenever  $N$  is big enough, that is, after many partitions the transformation of the moments is described (approximately) by (24) and  $\nu_{s+1} = \lambda_s$ . This observation allows us to obtain partition operator representation (7) from geometrical partition models, but formula (7) should be understood in some generalized sense.

A rigorous theory of the grinding equation can be found in [8], where it is shown that the grinding process can be correctly described in term of partition operators  $\mathcal{P}_{\Psi^*}$  and classifier operators  $\mathcal{C}_\alpha$  defined in spaces of generalized distribution functions. The dynamics of the grinding process is approximately described as  $\mathcal{P}^N(f) \approx \mathcal{P}_{\Psi^*}^N(F^*)$  whenever  $N$  is big enough. Here  $F^*$  is a generalized distribution function and  $\mathcal{P}_{\Psi^*}^N$  is a partition operator in the space of generalized distribution functions. This implies that the input and the output of a grinding system are related (approximately) by the grinding equation. The equation has a unique solution in an appropriate space. The study of geometric partition models, especially the study of the matrices  $P_s$  dominant eigenvalues, allows us to compute the partition operator  $\mathcal{P}_{\Psi^*}$ .

### 5.3. Possible application

Let us mention possible applications of the theory developed above. A grinding system can be composed of various grinders and classifiers. The grinding equation is an adequate mathematical model, which allows us to predict the results of grinding and construct grinding systems with desired properties.

An experimental verification of mathematical models describing grinding processes is rather difficult because different instruments used in the particle size measurement give different results [9]. Usually sizers interpret all particles as spheres independently of the particle shape. For example, if a sizer uses a laser light scattering, then the results of the scattered radiation measurements are interpreted as the far-field diffraction pattern of an assembly of spheres. The moment analysis can help to recover the ‘true’ particle size distributions from the results of the measurements.

Consider a particle of the volume  $V$  and the shape  $m$  having an orientation  $\omega \in \Omega$ , where  $\Omega$  is a set of possible orientations of particles in the sizer. Assume that the sizer interprets the particle as a ball of the volume  $T(m, \omega)V$ . Then the density function measured by the device is given by

$$f^b(V) = \sum_{m=1}^M \int_{\Omega} f_m \left( \frac{V}{T(m, \omega)} \right) h_m(\omega) d\omega,$$

where  $h_m$  are the density functions of the random value  $\omega$ , the orientation of a particle of the shape  $m$ . It is easy to calculate

$$\mu_s^b = \int_0^{\infty} V^s f^b(V) dV = \sum_{m=1}^M \int_{\Omega} \int_0^{\infty} V^s f_m \left( \frac{V}{T(m, \omega)} \right) dV h_m(\omega) d\omega = \sum_{m=1}^M \theta_{s+1}^{(m)} \mu_s^{(m)},$$

where

$$\theta_s^{(m)} = \int_{\Omega} T^s(m, \omega) h_m(\omega) d\omega.$$

The vector  $\bar{\mu}_s$  (see (23)) approximately has the form  $\bar{\mu}_s \approx \mu_s \hat{\mu}_s$ , where  $\hat{\mu}_s$  is an eigenvector of the matrix  $P_s$  with the coordinates  $\hat{\mu}_s^{(m)}$ ,  $m = \overline{1, M}$ , satisfying  $\sum_m \hat{\mu}_s^{(m)} = 1$ . Therefore

$$\mu_s^b = \mu_s \sum_{m=1}^M \theta_{s+1}^{(m)} \hat{\mu}_s^{(m)}.$$

From this we can find the moments  $\mu_s$  and estimate the density function  $f$ .

## 6. Example

Consider a two-dimensional powder composed of ellipses

$$E = \left\{ (x, y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}$$

of two types:  $a = b$  and  $a = 2b$  (figure 1). Assume that there are three possible orientations of the ellipses with respect to the plane (in this case the line)  $\Pi_0$ : the normal vector to  $\Pi_0$  is parallel to the axis  $Ox$ , to a line between the axes  $Ox$  and  $Oy$ , and to the axis  $Oy$ . The corresponding geometric partition model is described by the following rules:

$$\mathcal{R}(1, l) = (2, 2, 1), \quad l = \overline{1, 3},$$

$$\mathcal{R}(2, l) = \begin{cases} (1, 1, 1), & l = 1, \\ (2, 2, \gamma), & l = 2, \\ (2, 2, 1), & l = 3, \end{cases}$$



Figure 1. The shapes.

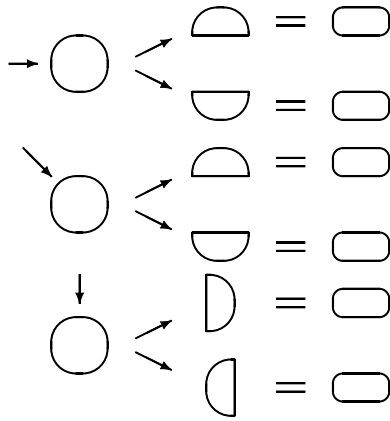


Figure 2. Partition rule for the first shape.

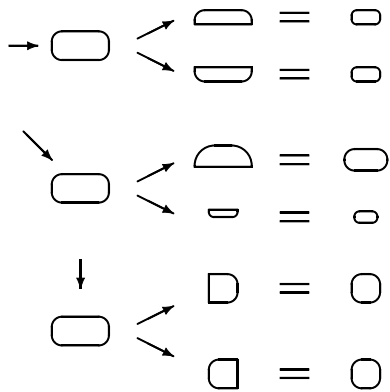


Figure 3. Partition rule for the second shape.

where  $\gamma \in ]0, 1[$  (figures 2 and 3). Transformation (20) of the functions  $f_1$  and  $f_2$  is given by

$$\mathcal{P}(f_1)(V) = \frac{1}{6}(4f_2(2V)),$$

$$\mathcal{P}(f_2)(V) = \frac{1}{6}(12f_1(2V) + 4f_2(2V) + (1 + 1/\gamma)f_2((1 + 1/\gamma)V) + (1 + \gamma)f_2((1 + \gamma)V)).$$

Formula (23) takes the form

$$\mathcal{P} \begin{pmatrix} \mu_s^{(1)} \\ \mu_s^{(2)} \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 0 & 1/2^{s-1} \\ 3/2^{s-1} & 1/2^{s-1} + (\gamma/(1 + \gamma))^s + 1/(1 + \gamma)^s \end{pmatrix} \begin{pmatrix} \mu_s^{(1)} \\ \mu_s^{(2)} \end{pmatrix}.$$

The function  $\psi$  reconstruction from the set of matrices  $P_s$  dominant eigenvalues is a hard ill-posed problem. The reconstruction can be even impossible if the sequence of the eigenvalue does not correspond to a function. However one can always apply the Tikhonov regularization

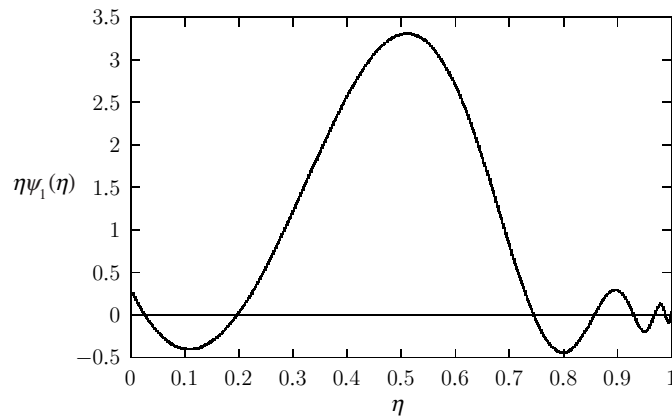


Figure 4. Two shapes, tree angles:  $\eta\psi_1(\eta)$ .

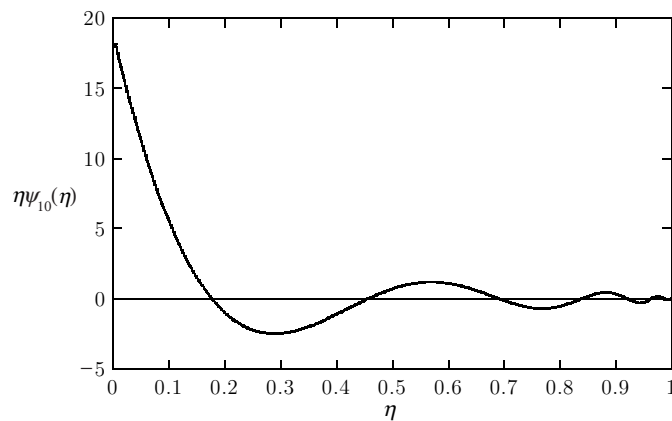


Figure 5. Two shapes, tree angles:  $\eta\psi_{10}(\eta)$ .

method [10, 11] to try to solve this problem. The results of such reconstruction for the functions  $\psi_\zeta$  (see (25)) corresponding to  $\zeta = 1$  and  $\zeta = 10$  are presented in figures 4 and 5, respectively. This model is far from reality and it is no wonder that the result does not satisfy natural physical requirement, namely the functions have negative values.

## 7. Comparison with experimental data

In order to verify the applicability of the geometric partition model approach, the computer simulated particle size distributions were compared with those produced by a laboratory mill with a sieve classifier. A laser diffraction based instrument (Malvern Mastersizer S) was used to measure particle size.

The geometric partition model includes ellipsoids

$$E = \left\{ (x, y, z) \in R^3 \left| \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \right. \right\}$$

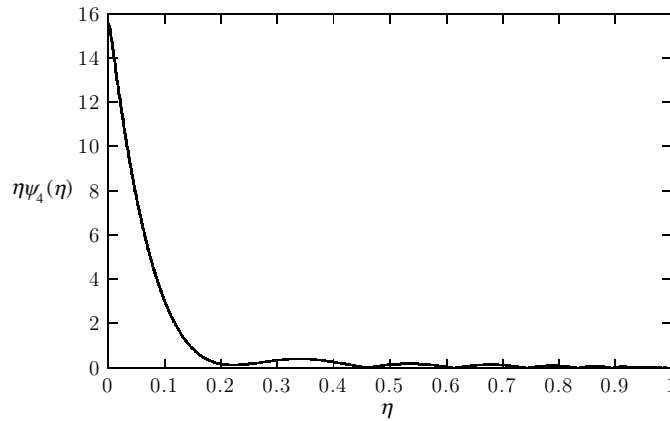


Figure 6. Twenty-eight shapes, eighty-one angles:  $\eta\psi_4(\eta)$ .

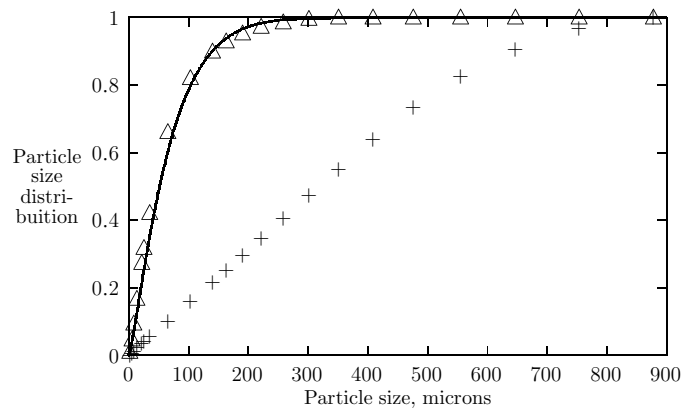


Figure 7. Material: epoxy resin. Distributions: '+' input, 'Δ' output, '-' simulated output.

of 28 types:  $a = 1, b = k/7, c = m/7, k = \overline{1,7}, m = \overline{k,7}$ . The number of possible orientations of the ellipsoids with respect to the plane  $\Pi_0$  is equal to 81. The parts  $\pi_1$  and  $\pi_2$  resulting from the breakup of an ellipsoid  $E = \pi_1 \cup \pi_2$  were approximated by ellipsoids  $E_1$  and  $E_2$  (with the same volumes  $V(E_r) = V(\pi_r), r = 1, 2$ ) in the sense of the Hausdorff distance:

$$h(\pi_r, E_r) = \min\{\rho \geq 0 \mid \pi_r \subset N_\rho(E_r), E_r \subset N_\rho(\pi_r)\}, \quad r = 1, 2.$$

Here  $N_\rho(A)$  stands for the  $\rho$ -neighbourhood of a set  $A$ . The function  $\psi_\zeta$  (see (25)) graph corresponding to  $\zeta = 4$  is presented in figure 6. The reconstruction of this function was fulfilled under the following additional condition:  $\psi_4(\eta) \geq 0, \eta \in [0, 1]$ . The results of the grinding process simulation for the classifier parameters  $\alpha = (\pi/6)10^6$  and  $\sigma = \alpha/3$  (see (9)) are presented in figures 7 and 8. The grinding equation was solved by the method of iterations, i.e. using (10). The parameters  $\zeta, \alpha$  and  $\sigma$  were chosen in order to fit the experimental data.

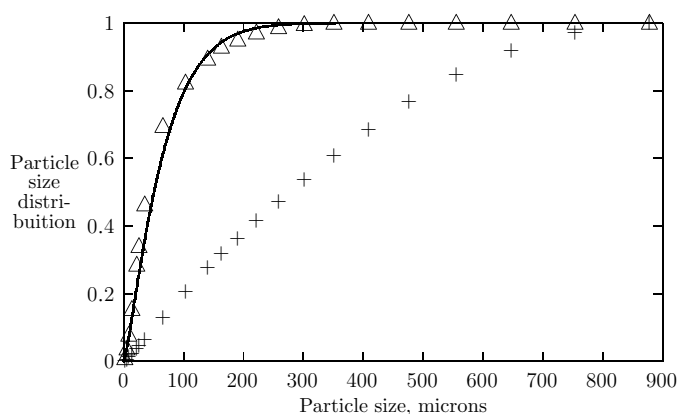


Figure 8. Material: polyester. Distributions: '+' input, 'Δ' output, '—' simulated output.

## 8. Conclusion

The geometric partition model approach developed to calculate the breakage function allows us to obtain a mathematical model of a mill with a classifier. The model contains three parameters  $\zeta$ ,  $\alpha$  and  $\sigma$  that are in our disposal. Adjusting these parameters one can get an adequate model of a grinding system composed of a grinder and a classifier.

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